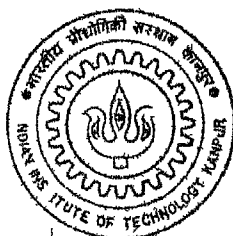# INSTRUMENTATION OF LOW FREQUENCY NOISE THROUGH WAVELETS

by

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DEPARTMENT OF ELECTRICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

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# INSTRUMENTATION OF LOW FREQUENCY NOISE THROUGH WAVELETS

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for the Degree of

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by

## AMIT SAXENA

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DEPARTMENT OF ELECTRICAL ENGINEERING

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# Certificate

This is to certify that the work contained in this thesis, entitled *INSTRUMEN-TATION OF LOW FREQUENCY NOISE THROUGH WAVELETS* has been carried out by *AMIT SAXENA* under our supervision and that this work has not been submitted elsewhere for a degree

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*Dedicated to*

*My Parents*

# ACKNOWLEDGEMENTS

# Abstract

Measurement of noise is important in engineering  A method to measure power spectral density (P S D ) of low frequency noise ($1/\mid f \mid^{\gamma}$ ) has been developed In this method "Constant Q" filtering of noise is done  For 'constant Q" filtering purpose Wavelet transform is used  Wavelet transform has perfect reconstruction and "Constant Q" properties  All cases of noises (different $\gamma$s ) are studied and algorithm is developed to give power spectral density of any type of noise  Perfect reconstruction property of Wavelet transform is used to synthesize $1/\mid f \mid^{1}$ noise from white noise

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# Chapter 1

# Introduction

Study of noise is of importance in science and engineering because it sets lower limits to the accuracy of any measurement  In order to evalute these limits one must know the magnitude and sources of noise involved and must learn how to measure and to locate these  Generally one uses two prototypes of noise  (1) White noise  and (2) flicker noise or coloured noise

   An easy way to distinguish between these noises is to consider their autocorrelation function $R_{xx}(\tau)$ as given by Keshner [1] and shown in fig 1 1
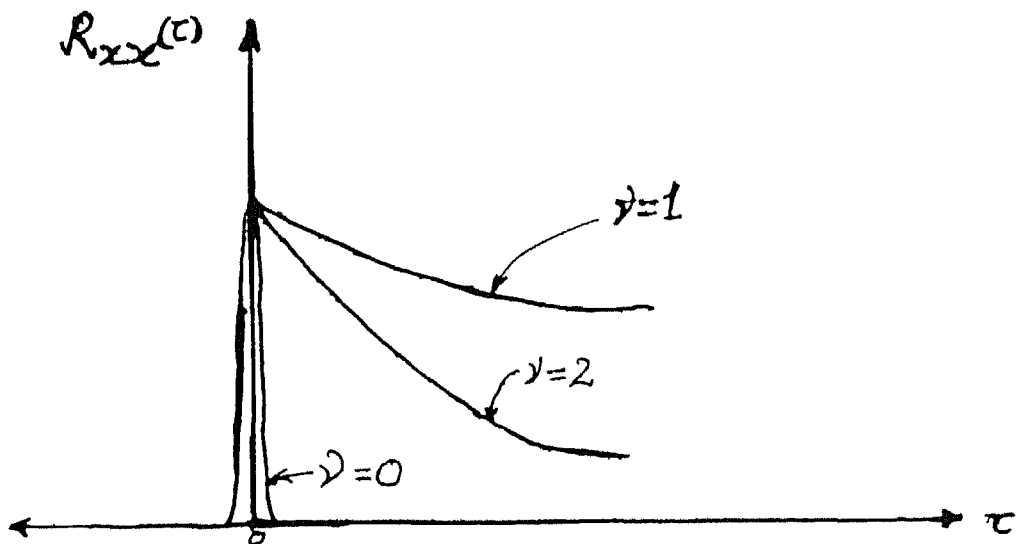


Figure 1 1  Autocorrelation function of 1/f noises

The white noise has the well known delta distributed autocorrelation function  If

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we represent the power spectral density of the noise as proprortional to $1/\mid f\mid^{\gamma}$ then the white noise case corresponds to $\gamma = 0$ The case of $\gamma = 2$ is also well known because a variety of processes give rise to exponentially decaying autocorrelation function, examples of these processes are random telegraph signals, generation recombination noise in semiconductor etc The case of $\gamma = 1$ is called ' 1/f noise ' or 'flicker noise' and has the largest autocorrelation as evident from fig 1 1 Moreover in engineering practice, one finds that ' 1/f noise is ubiquitous and appears in several physical phenomena Some examples of these are [1],[2]

1 noise in the currents and voltages of electronic devices ( e g field effect and bipolar transistors, schottky zener and tunnel diodes )

2 resistance fluctuations in metal film, semiconductor films and contacts ger manium filaments in carbon and aqueous solution thermocells and concen trations cells

3 burst errors in communication channels

4 physiological time series such as instantaneous heart rate records for healthy patients EEG variations under pleasing stimuli, and insulin uptake rate for diabetics

5 biological time series such as voltage across nerve and synthetic membranes

The study of flicker noise has been widely persued over almost six decades and even now several unresolved issues regarding it's modelling and it s affect on mea surement and communication are remaining The emphasis in the present work has been on the measurement issues and, particularly, an attempt has been made to exploit the recently developed wavelet theory [3] [8] for analysing and synthesizing 1/f noise as developed by Wornell [2] Our contribution in this work is in designing the filters and verifying the algorithms which relate to a theorem on 1/f noise as proved by Wornell [2] But before presenting this work, let us consider some simple schemes for the study of noise

Measurement of white noise can be done by constant bandwidth filters as shown in figure 1 2(a)  Here filters are of constant bandwidth, therefore power output of each filter will be the same  But if noise is $1/\mid f\mid^{\gamma}$ and $\gamma = 1$, then it's measurement can be done by "Constant Q"or Constant relative bandwidth filters as shown in fig 1 2(b)

WHITE
NOISE

1/f NOISE

a1
a2
a3
a4
a5

p1
p2
p3
p4
p5
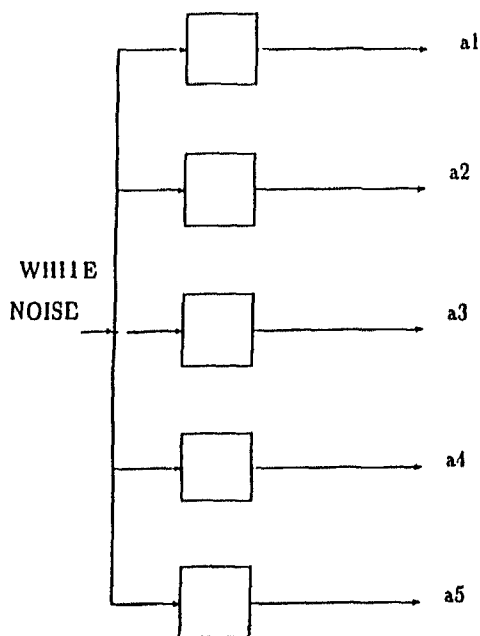
(a) CONSTANT BANDWIDTH FILTERS

a1=a2=a3=a4=a5

a and p are power output
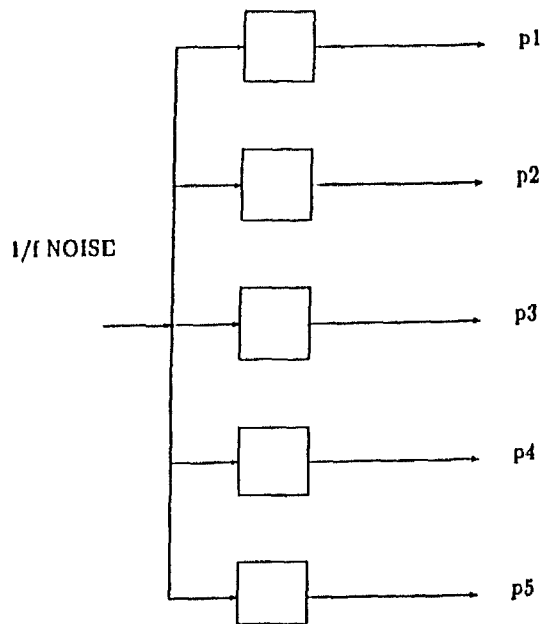
(b) CONSTANT Q FILTERS

p1=p2=p3=p4=p5

Figure 1 2  Constant Bandwidth And Constant Q Filters

Constant Q filters can be designed [14] through a number of existing methods other than that of wavelet based designing  We can get Constant Q filter bank by designing each filter individually  But this method will require a lot of computation Also with this method reconstruction will not be possible  Whereas in Constant

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Q filter designing through wavelets, only one sequence (which should fulfil certain conditions) is required Also this method ensures perfect reconstruction

In the present work "Constant Q" filters have been designed through wavelet theory These "Constant Q" filters are then used to give power spectral density of different noise sequences

To generate $1/f$ noise time series from white noise ( in order to analyse it by Constant Q filters)we have used two methods

- Through digital filters designed using wavlet theory

- Through digital filter obtained from analog filter using matched Z transforma tion for analog to digital conversion Analog filter has a slope of $-10db$ per decade

In first method a number of white noise sequences, having same power but differ ent lengths are required, whereas in second case only one white noise sequence is required

Because in practice we can't have ideal white noise sequence, we have used Knuth's [11] white noise sequence generator whose sequential correlation (shown in plot in ch 5) is very near to that of white noise Because of this fact, noise after filtering can not be $1/f$ in the whole range 0 to $\pi$

# 1 1 organization

This thesis is divided into six chapters

- In chapter 2 wavelet theory is discussed In this chapter decomposition and reconstruction algorithm for any sequence is given Decomposition algorithm decomposes a signal in to different sequences in which each sequence corre sponds to some frequency band of input sequence This therefore corresponds to Constant Q filtering of sequence

- In chapter 3, a theorem [2] is given, by which any type of noise can be synthe sized from white noise In this chapter results are drawn from this theorem ,

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to analyze any type of sequence through Constant Q filter bank

- In chapter 4 digital filter(s) design is given to generate $1/f$ noise Here both filter designing cases are discussed as stated earlier

- chapter 5 has results and relevant graphs

- In chapter 6 we have drawn conclusion/comparison based on above results

# Chapter 2

# Wavelet Theory

Wavelet theory provides a unified framework for a number of techniques which had been developed independently for various signal processing applications

In fact wavelet theory covers quite a large area  It treats both the continuous and discrete-time cases  It provides very general techniques that can be applied to many tasks in signal processing, and therefore has numerous potential applications  In particular, the wavelet transform is of interest for the analysis of non stationary signals, because it provides an alternative to the classical STFT (short time Fourier transform) or gabor transform  The basic difference between the two cases is in the choice of window[1]used  In STFT a simple analysis window is used, whereas in wavelet transform short windows at higher frequencies and long windows at lower frequencies are used  This is in the spirit of so called "Constant Q" or constant relative bandwidth frequency analysis

For some applications it is desirable to see the WT as a signal decomposition on to a set of basis functions  In fact, basis functions called WAVELETS always underlie the wavelet analysis  They are obtained from a single prototype wavelet by dilations and contractions as well as shifts  The prototype wavelet can be thought of as a bandpass filter, and the constant Q property of the other bandpass filters (wavelets) follows because they are scaled versions of the prototype

---

[1]By window we mean a signal having a window like structure which has negligible amplitude outside certain range

There are seveial types of wavelet transforms, and depending upon the applica tion, one may be preferred to the others For a continuous input signal, the time and scale parameters can be continuous, leading to a continuous Wavelet Trans form(CWT) They may as well be discrete leading to a wavelet series expansion Finally the wavelet transform can be defined for discrete time signals leading to a discrete WT In the latter case it uses *Multiresolution signal processig technique* In this thesis we have used *Multiresolution signal processing technique* as a tool to get information about P S D of $1/|f|^{\gamma}$ noise, so we will emphasize on this method

# 2 1 A Review Of Multiresolution Analysis And Orthonormal Wavelet Bases

The idea of multiresolution analysis [3] is to write $L^2$-functions $f$ as a limit of succesive approximations, each of which is a smoothed version of $f$, with more and more concentrated smoothing functions

More precisely, a multiresolution analysis consists of

(1) a family of closed embedded subspaces $V_m \subset L^2(R)^2$, $m \in Z^3$

$$\subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \tag{2 1}$$

such that (2)

$$\bigcap_{m \in Z} V_m = \{0\}, \quad \overline{\bigcup_{m \in Z} V_m} = L^2(R) \tag{2 2}$$

and (3)

$$f \in V_m \leftrightarrow f(2) \in V_{m-1}, \tag{2 3}$$

moreover, there is a $\phi \in V_0$ such that, for all $m \in Z$, the $\phi_{mn}$ constitute an unconditional basis for $V_m$, that is,(4)

$$\overline{V_m = linear\ span\{\phi_{mn}, n \in Z\}} \tag{2 4}$$

---

[2]Vector space of all measurable square integrable one dimensional functions

[3]Set of integers

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and there exit $0 < A \le B < \propto$ such that, for all $C_{nn \in Z} \in l^2(Z)$,

$$A \sum_n | C_n |^2 \le \| \sum_n C_n \phi_{mn} \|^2 \le B \sum_n | C_n |^2 \qquad (2\ 5)$$

Here $\phi_{mn} = 2^{-m/2} \phi(2^{-m}x - n)$  Let $P_m$ denote the orthogonal projection onto $V_m$ It is clear from  2 1,  2 2, that $lim_{m \to -\propto} P_m f = f$, for all $f \in L^2(R)$  The condition 2 3ensures that $V_m$ correspond to different scales, while the translational invariance

$$f \in V_m \to f( - 2^m n) \in V_m \text{ for all } n \in Z \text{ is a consequence of } 2\ 4 \text{ and } 2\ 5 \quad \phi(x)$$

is known as scaling function or father wavelet

Defining

$$C_{mn}(f) \quad = \quad < \phi_{mn}, f > \qquad (2\ 6)$$

$$Then \quad P_m f \quad = \quad \sum_n C_{mn}(f) \phi_{mn} \qquad (2\ 7)$$

$P_m f$ is the approximated version of $f$ in $V_m$ vector space  Now again if we ap proximate f at $V_{m-1}$ then the information loss between two approximations can be obtained from projection theorem which is equal to the orthogonal projection onto orthogonal complement of $V_m$ in $V_{m-1}$ ı e  defining $W_m$ as the orthogonal comple ment, in $V_{m-1}$, of $V_m$,

$$V_{m-1} = V_m \oplus W_m \qquad , \quad W_m \perp V_m \qquad (2\ 8)$$

Equivalantely,

$$W_m = Q_m L^2(R) \quad with \quad Q_m = P_{m-1} - P_m \qquad (2\ 9)$$

It follows immediately that all the $W_m$ are scaled versions of $W_0$

$$f \in W_m \leftrightarrow f(2^m ) \in W_0 \qquad (2\ 10)$$

and that the $W_m$ are orthogonal spaces which sum to $L^2(R)$

$$L^2(R) = \bigoplus_{m \in Z} W_m \qquad (2\ 11)$$

Wavelet theory ensures (eq   2 1   eq   2 5) that there exist $\psi \in W_0$ such that $\psi_{mn}$ contitutes unconditional bases for $W_m$

This completes our analysis

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To start our analysis we choose $\phi$ satisfying 2 1   2 5, then we get $\psi(x)$ from $\phi(x)$ through following procedure  Since $\phi \in V_0 \rightarrow \phi(2x - n) \in V_{-1}$, also since $V_0 \subset V_{-1}$ hence $\phi \in V_{-1}$  Thus there will exist $C_n$ such that,

$$\phi(x) = \sum_n C_n \phi(2x - n) \tag{2 12}$$

Then we define

$$\psi(x) = \sum_n (-1)^n C_{n+1} \phi(2x + n) \tag{2 13}$$

$\psi$ is called mother wavelet and $\psi_{mn}$ are dilated(or contracted) and scaled versions of mother wavelet

## 2 3   Wavelet based decomposition and reconstruction algorithm

Let us assume that we have a sequence $C_n^0$  We want to decompose this signal in levels correspondoing to different frequency bands of input sequence

To do this we will use multiresolution analysis, i e we choose $\phi$ satisfying 2 1

2 5  After using this decomposition algirithm, sequences obtained will correspond to different frequency bands of input sequence  In other words this algorithm will correspond to passing signal through Constant Q filters

### 2 3 1   decomposition algorithm

From $C_n^0$ we form a function f as follows

$f = \sum_n C_n^0 \phi_{0n}$, or

$f(x) = \sum_n C_n^0 \phi(x - n)$,

since $V_0$ is linear span of set $\{\phi_{0n}, n \in Z\}$ hence $f \in V_0$

Also from our construction $V_0 = V_1 \oplus W_1$, hence we can write,

$$f = P_1 f + Q_1 f \tag{2 14}$$

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where $\phi_{1n} \rightarrow$ bases for $V_1$, and $\psi_{1n} \rightarrow$ bases for $W_1$ hence from 2 6 and 2 7 we can write,

$$P_1 f = \sum_n C_{1n}(f)\phi_{1n} \ , \ \ C_{1n}(f) = < \phi_{1n} \ f > \tag{2 15}$$

Where $P_1 f$ is approximation of $f$ in $V_1$ vector space

Using similer technique we can write for $Q_1 f$

$$Q_1 f = \sum_n d_{1n}(f)\psi_{1n} \ , \ \ d_{1n}(f) = < \psi_{1n} \ f > \tag{2 16}$$

Where $Q_1 f$ is difference of information between f and it's approximated version $P_1 f$

Writing $c_n^1$ for $C_{1n}(f)$ and $d_n^1$ for $d_{1n}(f)$

From 2 15

$$
\begin{aligned}
c_n^1 &= \ < \phi_{1n}, f > \\
c_n^1 &= \ \int 2^{-1/2}\phi(2^{-1}x - n)f(x)dx \\
c_n^1 &= \ \int 2^{-1/2}\phi(2^{-1}x - n)\sum_k c_k^0\phi_{0k}dx \quad from \ above \\
c_n^1 &= \ \sum_k c_k^0 \int 2^{-1/2}\phi(2^{-1}x - n)\phi(x - k)dx \\
c_n^1 &= \ \sum_k 2^{-1/2}c_k^0 \int \phi(x/2)\phi(x - (k - 2n))dx \\
c_n^1 &= \ \sum_k h(k - 2n)c_k^0 \tag{2 17}
\end{aligned}
$$

Where $h(n) = 2^{-1/2}\int \phi(x/2)\phi(x-n)dx$ , is known as lowpass analysis filter Looking at eq 2 17 it is clear that $c_n^1$ from $c_n^0$ can be obtained by filtering it through $h(-n)$ and then subsampling it by 2 Similarly for $d_n^1$, from eq 2 16,

$$
\begin{aligned}
d_n^1 &= \ < \psi_{1n} \ f > \\
d_n^1 &= \ < \psi_{1n} \ \sum_k c_k^0\phi_{0k} > \quad from \ above \\
d_n^1 &= \ \sum_k c_k^0 < \psi_{1n} \ \phi_{0k} > \\
d_n^1 &= \ \sum_k c_k^0 2^{-1/2} \int \psi(x/2 - n)\phi(x - k)dx \\
d_n^1 &= \ \sum_k c_k^0 2^{-1/2} \int \psi(x/2)\phi(x - (k - 2n))dx \\
d_n^1 &= \ \sum_k c_k^0 g(k - 2n) \tag{2 18}
\end{aligned}
$$
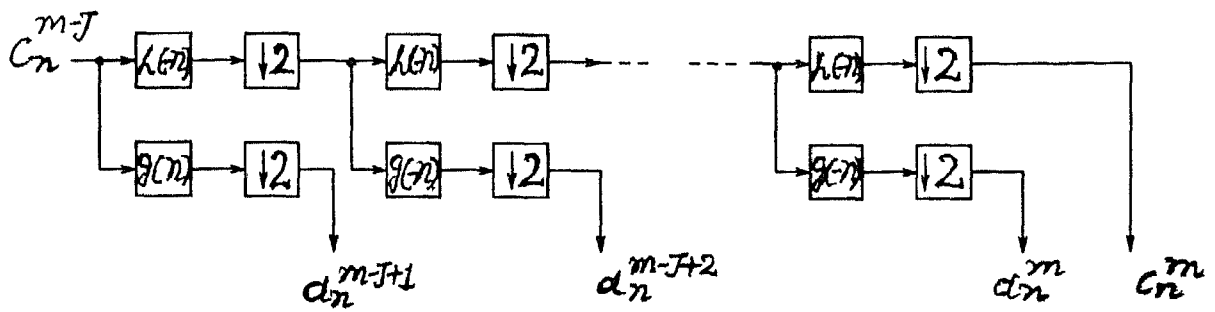
(a)DECOMPOSITION ALGORITHM
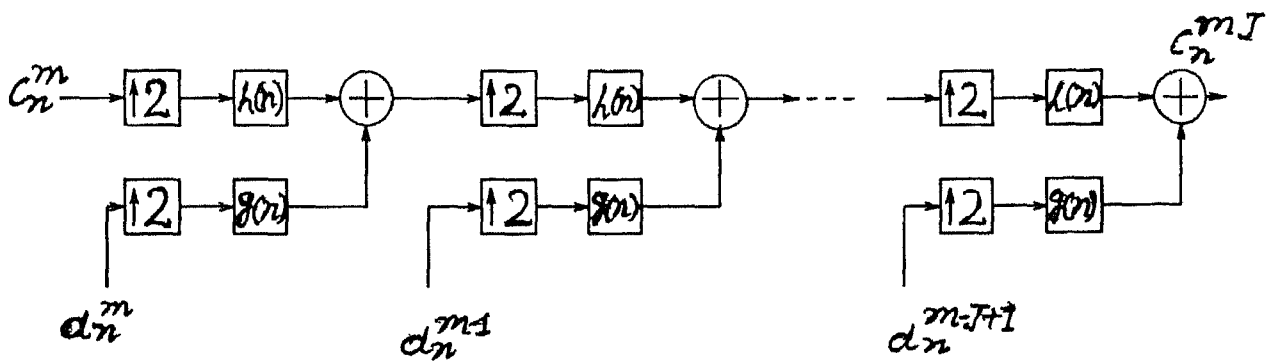


(b)RECONSTRUCTION ALGORITHM

Figure 2 1 Decomposition and reconstruction algorithm

where $g(n) = 2^{-1/2} \int \psi(x/2)\phi(x-n)dx$, is known as highpass analysis filter Looking at eq 2 18 and fig 2 1(a) it is clear that $d_n^1$ from $c_n^0$ can be obtained by filtering it through $g(-n)$ and then subsampling it by 2 After getting $c_n^1$ from $c_n^0$ we can again apply the same analysis on $c_n^1$ to get $c_n^2$ and $d_n^2$, i e we can get lower resolution signals from higher resolution signals by repeating the analysis algorithm just described This completes the analysis part

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## 2 3 2   Reconstruction algorithm

In reconstruction part (see figure 2 1) we have details($d_n^j$ ,j$\in [1, j]$) and one approx imated signal($c_n^j$) and we have to construct original signal $c_n^0$

We start with lowest resolution i e from $j^{th}$ resolution, and construct a function $\alpha$ as shown below,

$$\alpha = \underbrace{\sum_n c_n^j \phi_{jn}}_{a} + \underbrace{\sum_n d_n^j \psi_{jn}}_{b} \tag{2 19}$$

Clearly $a \in V_j$ and $b \in W_j$, hence $\alpha \in V_{j-1}$ , and we can write $\alpha = P_{j-1}f$

$$\alpha = P_{j-1}f = \sum_n c_n^{j-1} \phi_{j-1n} \quad from \quad eq \quad 2 7 \tag{2 20}$$

$$P_{j-1}f = \sum_n c_n^j \phi_{jn} + \sum_n d_n^j \psi_{jn} \tag{2 21}$$

Also

$$c_n^{j-1} = < \phi_{j-1n}, P_{j-1}f > \tag{2 22}$$

$$c_n^{j-1} = < \phi_{j-1n}, \sum_k c_k^{j-1} \phi_{j-1k} > from \; 2 20$$

$$= < \phi_{j-1n}, \sum_k c_k^j \phi_{jk} + \sum_k d_k^j \psi_{jk} > from \; 2 20$$

$$= \sum_k c_k^j < \phi_{j-1n} \; \phi_{jk} > + \sum_k d_k^j < \phi_{j-1n}, \psi_{jk} >$$

$$= \sum_k c_k^j 2^{-1/2} \int \phi(x/2)\phi(x - (n-2k))dx + \sum_k d_k^j 2_{-1/2} \int \psi(x/2)\phi(x - (n-2k))dx$$

$$c_n^{j-1} = \sum_k c_k^j h(n-2k) + \sum_k d_k^j g(n-2k) \tag{2 23}$$

i e  we have got $c_n^{j-1}$ from $c_n^j$ and $d_n^j$

Thus recursively using above algorithm we can get $c_n^0$  Reconstruction algorithm is shown in figure 2 1(b) in a schematic fashion

This completes decomposition and reconstruction algorithm which decomposes signal in to different frequency band signals and given these different frequency band signals, original signal can be constructed

To be more familiar with decomposition, reconstruction algorithms and detail and approximated signals we look at fig 2 1, and consider frequency spectra of these signals

Consider decomposition algorithm in which from $c_n^0$ (which will have frequency spectrum from 0 to $\pi$ ), first we get $c_n^1$ and $d_n^1$, where $c_n^1$ is obtained by filtering $c_n^0$ through h( n) and then subsampling it by 2  $d_n^1$ is obtained by filtering $c_n^0$ through g( n) and then subsampling it by 2  Here h( n) is half band low pass filter(pass band is 0 to $\pi/2$), and g( n) is half band high pass filter(pass band is $\pi/2$ to $\pi$)   Hence $c_n^1$ will have information from 0 to $\pi/2$ of $c_n^0$ and $d_n^1$ will have information from $\pi/2$ to $\pi$ of $c_n^0$  Again $c_n^2$ and $d_n^2$ are obtained from $c_n^1$ through similar procedure  $c_n^2$ will have information from 0 to $\pi/2$ of $c_n^1$ and $d_n^1$ will have information from $\pi/2$ to $\pi$ of $c_n^1$  In other words $c_n^2$ will have information from 0 to $\pi/4$ of $c_n^0$ and $d_n^2$ will have information from $\pi/4$ to $\pi/2$ of $c_n^0$

Hence after applying decomposition algorithm on a given signal $(c_n^0)$,$l$ times, we shall be having $(l+1)$ signals, in which $l$ signals will be detail signals(from $d_n^1$ to $d_n^l$) and one will be approximated signal($c_n^l$)

Here $d_n^1$ will correspond to $\pi/2$ to $\pi$ of $c_n^0$

Here $d_n^2$ will correspond to $\pi/4$ to $\pi/2$ of $c_n^0$

Here $d_n^3$ will correspond to $\pi/8$ to $\pi/4$ of $c_n^0$


Here $d_n^l$ will correspond to $\pi/2^{l-1}$ to $\pi/2^l$ of $c_n^0$

And $c_n^l$ will correspond to 0 to $\pi/2^l$ of $c_n^0$

Hence we have divided original signal in to $(l+1)$ signals correspondig to different frequency bands of original signal  Moreover what we have achieved through this decomposition is  good frequency resolution at low frequencies and poor frequency resolution at higher frequencies  (That is why it is called multiresolution analysis)

We can see two graphs given in chapter 5 right now  in which White and colour noises are decomposed in to different $d_n$'s and $c_n$ by the filters designed in chapter 4  Looking at $c_n^1$ and $d_n^1$ (mentioned in these graphes as 1/2 resolution signals) one can very easi ly see that in case of white noise both $c_n^1$ and $d_n^1$ are rich in magnitude, showing that $c_n^0$ has good spectrum over the entire range 0 to $\pi$  Whereas in colour noise case $d_n^1$ is very poor in magnitude as compared to $c_n^1$, showing that $c_n^0$ has very small spectrum in $\pi/2$ to $\pi$, which indeed is the fact

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We have discussed types of WT in the begining of this chapter There was a wavelet series expansion represantation of signal We will discuss this representation in next section

## 2 4   Wavelet series expansion representation

An orthonormal wavelet transformation of a signal $f(x)$ is generally described in terms of synthesis/analysis equations [2]

$$f(x) = \sum_m \sum_n d_n^m \psi_{mn}(x) \qquad (2\ 24)$$

where

$$
\begin{aligned}
d_n^m &= \ <\psi_{mn}, f> \\
&= \ \int f(x)\psi_{mn}(x)dx
\end{aligned}
\qquad (2\ 25)
$$

Considering synthesis equation i e eq 2 25 we can write

$$d_n^m = \int f(x)2^{-m/2}\psi(2^{-m}x - n)dx \qquad (2\ 26)$$

putting $2^{-m}x - n = 2^{-m}y$,

$$
\begin{aligned}
d_n^m &= \ \int f(y + 2^m n)2^{-m/2}\psi(2^{-m}y)dy \\
&= \ \int f(y + 2^m n)\psi_{m0}(y)dy \\
d_n^m &= \ \{f(x) * \psi_{m0}(-x)\}_{|x=2^m n}
\end{aligned}
\qquad (2\ 27)
$$

This representation is known as *octave band filter band interpretation* Thus for each m, $d_n$ can be obtained from *filter and sample operation* This representation is shown in figure 2 2

Considering analysis equation i e  eq  2 24, it can be viewed as multirate modu lation as depicted in figure 2 2

To undeistand how we get multirate modulation diagram as shown in fig 2 2 consider $d_n^m$, from eq 2 27, it is clear that for each m, $d_n^m$ correspond to samples with consecutive samples $2^m$ seconds apart
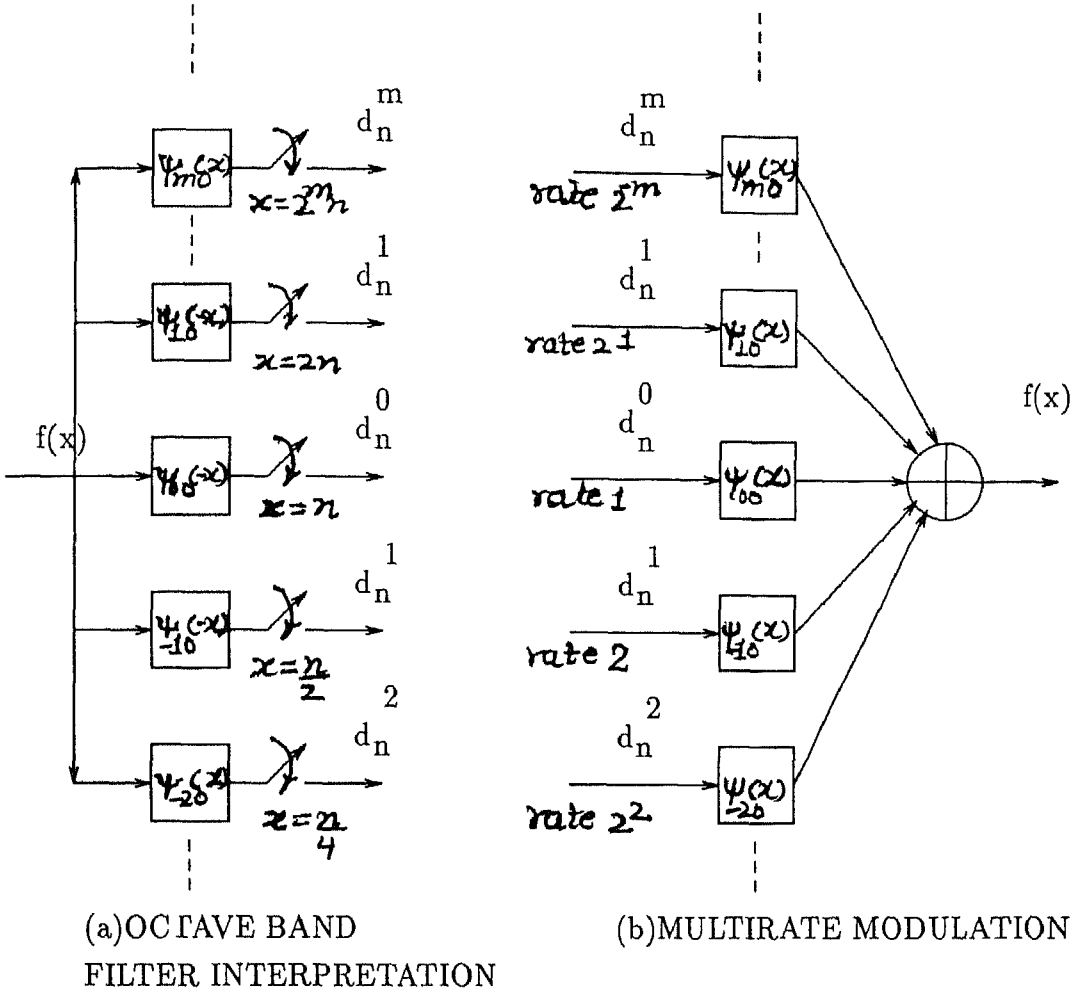
Figure 2 2   Octave band filter bank and multirate modulation interpretation

Hence $d_n^m$ will have form as shown in fig 2 3   And $d_n^m$ can be written as

$$d_n^m = \sum_{n_0} d_{n_0}^m \delta(n - n_0) \qquad (2\ 28)$$

where $n = 2^{-m}x$ from octave band filter representation, therefore

$$
\begin{aligned}
d_n^m * \psi_{m0}(x) &= \Big(\sum_{n_0} d_{n_0}^m \delta(2^{-m}x - n_0)\Big) * \psi_{m0}(x) \\
&= \sum_{n_0} d_{n_0}^m \big(\delta(2^{-m}x - n_0) * \psi_{m0}(x)\big) \\
&= \sum_{n_0} d_{n_0}^m \psi_{m0}(x - 2^m n_0)
\end{aligned}
$$

$m$

$2$ seconds

$n$
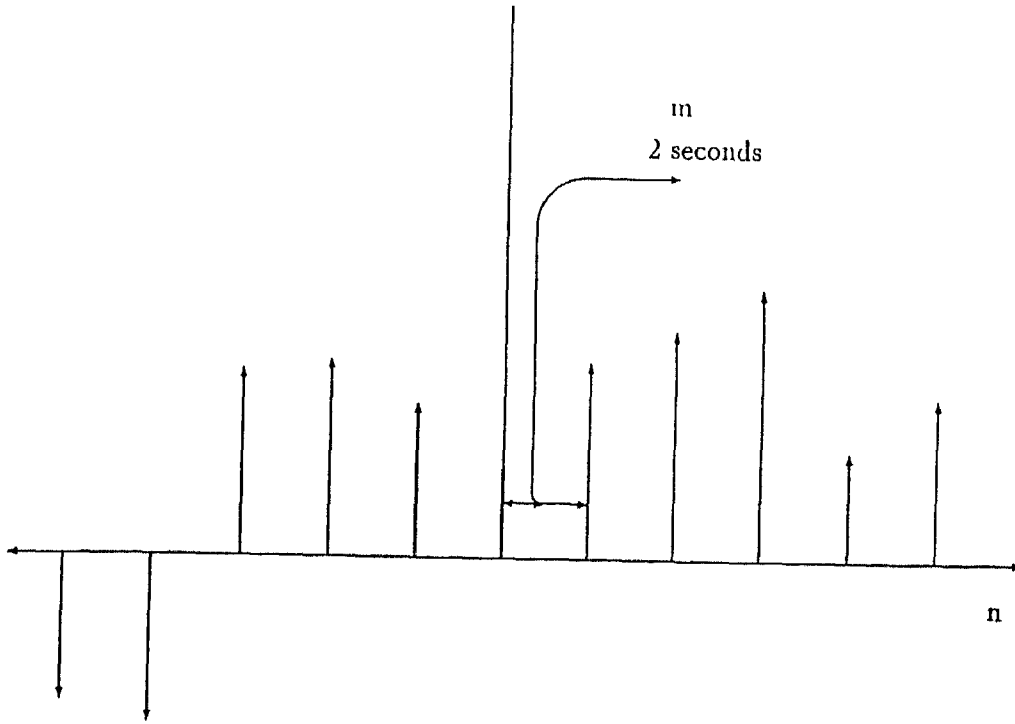
Figure 2 3   Sequence obtained from filter and sample operation

$$d_n^m * \psi_{m0}(x) \;=\; \sum_n d_n^m \psi_{m0}(x - 2^m n)$$

$$=\; \sum_n d_n^m 2^{-m/2} \psi(2^{-m}(x - 2^m n))$$

$$=\; \sum_n d_n^m 2^{-m/2} \psi(2^{-m} x - n)$$

$$d_n^m * \psi_{m0}(x) \;=\; \sum_n d_n^m \psi_{mn}(x)$$

or we can write

$$\sum_m (d_n^m * \psi_{m0}(x)) \;=\; \sum_m \sum_n d_n^m \psi_{mn}(x) \qquad (2\ 29)$$

$$=\; f(x) \qquad (2\ 30)$$

ı e   synthesis equation can be viewed as multirate modulation as shown ın figure 2 2

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# Chapter 3

# Synthesis And Analysis of Low Frequency Noise

In chapter 2 we have discussed decomposition algorithm, which decomposes a signal in to different frequency bands This algorithm corresponds to Constant Q filtering of input signal

Now we will state a theorem [2] by which any type of low frequency noise can be synthesized through reconstruction algorithm of chapter 2 This theorem also helps in making the choice of wavelet basis that should be used to synthesize a particuler noise Initially we will motivate the structure of this theorem and then a rigereous statement will be made

In chapter 2 we discussed wavelet based series expansion Let us write the function $f(x)$ as follows

$$f(x) = \sum_m \sum_n d_n^m \psi_{mn}(x), \tag{3 1}$$

Now in order to represent a process, having power spectral density of the form $\sigma^2 / \mid f \mid^\gamma$, using equation 3 1 constraints have to be imposed on $d_n^m$ and $\psi(x)$ These constraints are

1  $d_n^m$ should be a collection of mutually uncorrelated zero mean random variables with variances [2]

$$var\{d_n^m\} = \sigma^2\{1/L_m\}^\gamma \tag{3 2}$$

Here $L_m$ is length of sequence $d_n^m$ Note that $L_m = 2^{-m}$ from fig 2 3 therefore

$$\text{var}\{d_n^m\} = \sigma^2 2^{m\gamma}$$

2 Orthonormal wavbelet basis $\psi(x)$ should be having at least $\gamma/2$ order regularity[1]

Now we will state the theorem [2]

**THEOREM  3 1** *Consider any orthonormal wavelet basis with $R^{th}$ order regularity for some $R \geq 1$  Then the random process constructed via the expansion,*

$$f(x) = \sum_m \sum_n d_n^m \psi_{mn}(x)$$

*where $d_n^m$ are a collection of mutullay uncorrelated zero mean random variables with variances,*

$$var\ \{d_n^m\} = \sigma^2 2^{m\gamma},$$

*for some parameter $2R > \gamma > 0$ has a time averaged spectrum*

$$S_f(\omega) = \sigma^2 \sum_m 2^{\gamma m}|\ \psi(2^m\omega)\ |^2$$

*that is nearly $1/f$ i e*

$$\frac{\sigma_L^2}{|\omega|^\gamma} \leq S_f(\omega) \leq \frac{\sigma_u^2}{|\omega|^\gamma}$$

*for some $0 < \sigma_L^2 \leq \sigma_U^2 < \propto$ , and has octave spaced ripple i e  for any integer*

$K$

$$|\ \omega\ |^\gamma S_f(\omega) = |\ 2^K\omega\ |^\gamma S_f(2^K\omega)$$

With the help of this theorem, an algorithm is developed in the present work  to synthesize and analyze noises for different $f$s

Before proceeding towards next section we simplify eq  3 2 to get an equation for power($P_n$) of sequence $d_n^m$ as follows

$$var\{d_n^m\} = \sigma^2\{1/L_m\}^\gamma, \tag{3 3}$$

Note that

$$var\{d_n^m\} = \frac{1}{L_m}\sum_{n=1}^{L_m}\{d_n^m\}^2 \tag{3 4}$$

---

[1]see Appendix A

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Hence power($P_m$),

$$P_m = \sum_{n=1}^{L_m} \{d_n^m\}^2 = \sigma^2 \{1/L_m\}^{\gamma-1} \tag{3 5}$$

or we can write,

$$P_m = \sigma^2 2^{-m(1-\gamma)} \tag{3 6}$$

# 3 1   Synthesis of $1/\mid f \mid^\gamma$ noise

To synthesize $1/\mid f \mid^\gamma$ noise for given $\sigma^2$, $\gamma$, frequency range and output sequence length $L_{out}$, steps that should be followed are given in fig 3 1   Frequency range should be of the form $\pi$ to $\pi/2^k$, where $k \in Z$   Also $L_{out}$ should be a multiple of 2

To understand synthesis more clearly we consider an example  Suppose we want a sequence of length 1024 (It is a multiple of 2) with P S D $\sigma^2/\mid f \mid^\gamma$ over frequency range $\pi$ to $\pi/8$   In other words we have specified the values of $\sigma^2$, $\gamma$, frequency range and output sequence length, and using above theorem we want to construct a sequence, (from white noise sequences) having P S D  very near to as desired

For this example $k = 3$ (see fig 3 1) and hence

1   Three detail signals are required

   (a) $\pi/2$ to $\pi$ signal

   (b) $\pi/4$ to $\pi/2$ signal

   (c) $\pi/8$ to $\pi/4$ signal

2   Length of $\pi/8$ to $\pi/4$ signal is

   $L_{out}/2^k = 1024/8 = 128$

   Since $L_m = 2^{-m}$ therefore $m = -7$ for $\pi/8$ to $\pi/4$ signal  Hence this signal will be represented as $d_n^{-7}$

   Similarly length of $\pi/4$ to $\pi/2$ ($d_n^{-8}$) signal $= 256$

   And length of $\pi/8$ to $\pi/4$ ($d_n^{-9}$) signal $= 512$

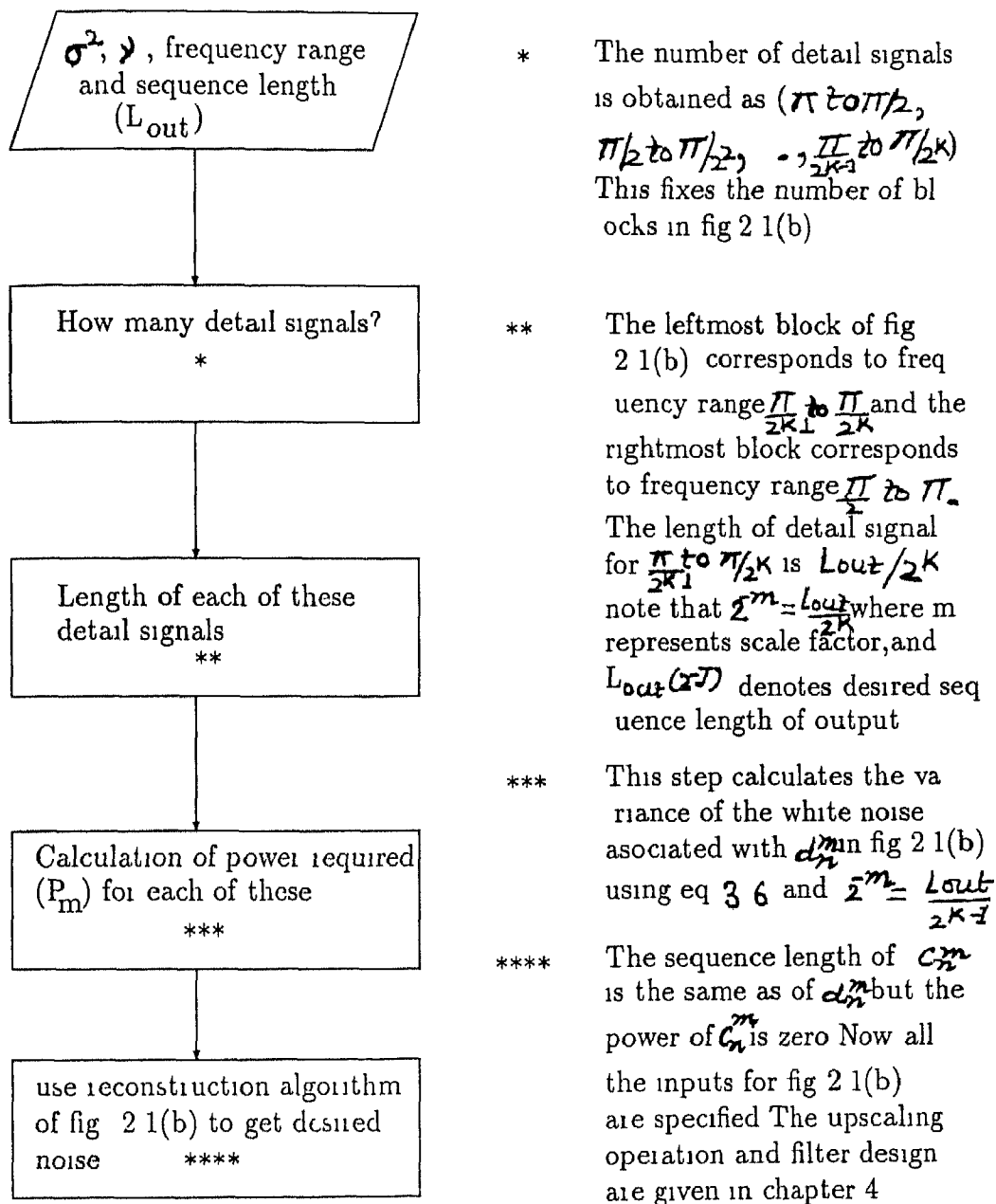3   Power required of $d_n^{-7}$ signal $= P_{-7} = \sigma^2 2^{-7(1-\gamma)}$ from eq  3 6

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$\sigma^2, \lambda$ , frequency range and sequence length $(L_{out})$

$\downarrow$

How many detail signals?

*

$\downarrow$

Length of each of these detail signals

**

$\downarrow$

Calculation of power required $(P_m)$ for each of these

***

$\downarrow$

use reconstruction algorithm of fig 2 1(b) to get desired noise    ****

---

* The number of detail signals is obtained as $(\pi \; to \; \pi/2,$ $\pi/2 \; to \; \pi/2^2, \; \cdots, \frac{\pi}{2^{K-1}} \; to \; \pi/2^K)$ This fixes the number of blocks in fig 2 1(b)

** The leftmost block of fig 2 1(b) corresponds to frequency range $\frac{\pi}{2^{K-1}}$ to $\frac{\pi}{2^K}$ and the rightmost block corresponds to frequency range $\frac{\pi}{2}$ to $\pi$. The length of detail signal for $\frac{\pi}{2^{K-1}}$ to $\pi/2^K$ is $L_{out}/2^K$ note that $2^m = \frac{L_{out}}{2^K}$ where m represents scale factor, and $L_{out}(x)$ denotes desired sequence length of output

*** This step calculates the variance of the white noise asociated with $d_n^m$ in fig 2 1(b) using eq 3 6 and $2^m = \frac{L_{out}}{2^{K-1}}$

**** The sequence length of $c_n^m$ is the same as of $d_n^m$ but the power of $c_n^m$ is zero Now all the inputs for fig 2 1(b) are specified The upscaling operation and filter design are given in chapter 4

Figure 3 1 Steps for arranging the power and sequence length of input $d_n^m$ and $c_n^m$ in fig 2 1(b)

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Power required of $d_n^{-8}$ signal $= P_{-8} = \sigma^2 2^{-8(1-\gamma)}$

Power required of $d_n^{-9}$ signal $= P_{-9} = \sigma^2 2^{-9(1-\gamma)}$

Hence by using $d_n^{-7}$, $d_n^{-8}$ and $d_n^{-9}$, we can get $c_n^{-10}$ through reconstruction algorithm which will have P S D very near to $\sigma^2/ \mid f \mid^\gamma$ in the range $\pi/8$ to $\pi$

From above we have concluded following

1 To synthesize $1/ \mid f \mid^\gamma$ noise for $1 > \gamma \geq 0$, lower resolution signals (higher values of m) must have lesser power than higher resolution signals (see eq 3 6) In other words to synthesize $1/ \mid f \mid^\gamma$ noise for $\gamma < 1$ case we require zero mean uncorrelated sequences of different lengths As length of sequence increases(resolution increases) power of sequence must increase

2 To synthesize $1/ \mid f \mid^\gamma$ noise for $\gamma = 1$ case, for all resolution signals, power required is same In other words to synthesize $1/ \mid f \mid^1$ noise we require zero mean uncorrelated sequences of different lengths but of same power

3 To synthesize $1/ \mid f \mid^\gamma$ noise for $1 < \gamma$, lower resolution signals (higher values of m) must have higher power than higher resolution signals (see eq 3 6) In other words to synthesize $1/ \mid f \mid^\gamma$ noise for $\gamma > 1$ case, we require zero mean uncorrelated sequences of different lengths As length of sequence increases(resolution increases) power of sequence must decrease

## 3 2 Analysis of $1/ \mid f \mid^\gamma$ noise

In analysis part we are given a sequence and $\sigma^2$ and $\gamma$ of this sequence has to be found out This is done through "Constant Q" filter bank designed using wavelet transform (See fig 2 1(a)) Once we are given a sequence, and it has been passed through decomposition filters ("Constant Q" filters), we shall be having one approximated and other detail signals These signals will be of varying length with each decrease in resolution (increase in value of m), length will decrease to approximately half of its previous length Hence after passing given sequence through decomposition filters, we shall have $P_m$ and $L_m$ of each detail signal, and we want to know $\sigma^2$

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and $\gamma$  From eq 3 5 we have

$$P_m = \sigma^2 \{1/L_m\}^{\gamma-1} \tag{3 7}$$

or

$$\log_2 P_m = \log_2 \sigma^2 + (1 - \gamma)\log_2 L_m \tag{3 8}$$

Looking at above equation, we can make following conclusions regarding $\gamma$ of input sequence from the knowledge of $P_m$ , $L_m$  and the fact that as resolution decreases length of detail signal decreases  These conclusions are

1  If power obtained in coarser detail signal is less than power obtained in finer detail signal, it is a $\gamma < 1$ case

2  If power obtained in each detail signal is same then its a $\gamma = 1$ case

3  If power obtained in coarser detail signal is more than power obtained in finer detail signal, it is a $\gamma > 1$ case

(** We could reach to the same conclusions as stated above in 1, 2, and 3 by looking at fig 3 2  In this figure power spectra of $\gamma < 1$  $\gamma = 1$ and $\gamma > 1$ noises are shown It is evident from this figure that power spectrum of $\gamma > 1$ noise converges faster than $\gamma = 1$ noise, whereas power spectrum of $\gamma < 1$ noise converges slower than $\gamma = 1$ noise

We are using "Constant Q" filters to analyse noise and if input noise has $\gamma = 1$ then power output of each of these filter will be the same  But because of faster convergence of $\gamma > 1$ noise at higher frequencies (higher resolution)  power output of higher resolution signal will we lesser than power output of lower resolution sig nals  Similarly because of slower convergence of $\gamma < 1$ noise at higher frequencies power output of higher resolution signal will be higher than that of lower resolution signal **)

Therefore by plotting a graph between $\log L_m$ and $\log p_m$ for different detail signals obtained from "Constant Q  filter bank, we can find $\sigma^2$ and $\gamma$  In next section we have considered an example for analysis of $1/\mid f \mid^\gamma$ noise

$S(\omega)$  $\pi/4$  $\pi/2$  $\pi$  $\omega$

$\gamma < 1$

$S(\omega)$  $\pi/4$  $\pi/2$  $\pi$  $\omega$

$\gamma = 1$

$S(\omega)$  $\pi/4$  $\pi/2$  $\pi$  $\omega$
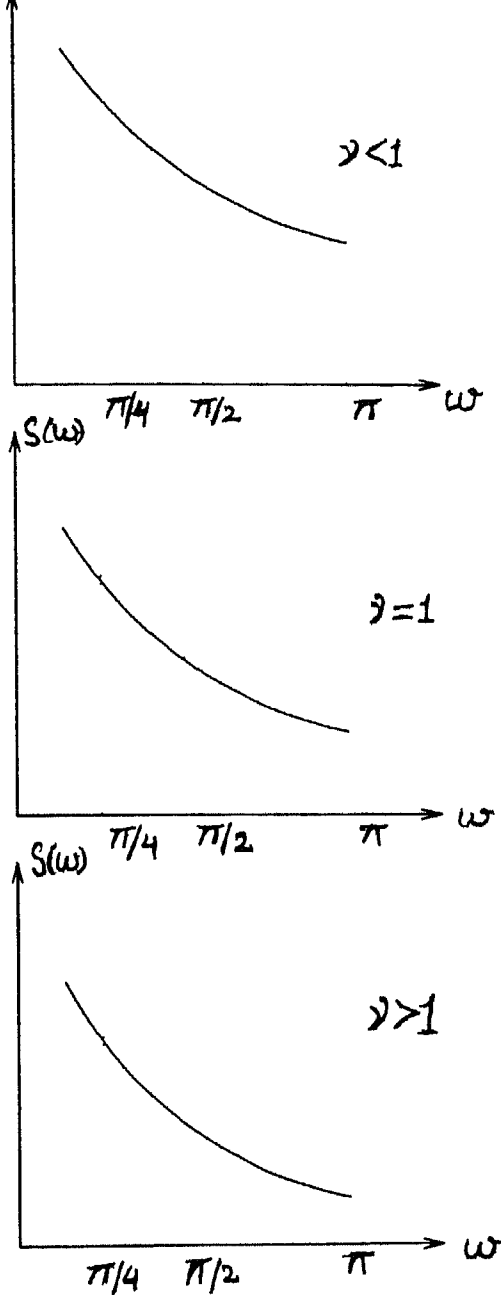
$\gamma > 1$

Figure 3 2  Power spectra of $1/\mid f \mid^{\gamma}$ noise

## 3 2 1  Example

Let us assume that we have a sequence of length 1024, and we want to know value of $\sigma^2$ and $\gamma$ over certain frequency range  Here frequency range will decide how many times decomposition algorithm has to be applied on the input signal  If we want to know $\sigma^2$ and $\gamma$ over frequency range ($\pi$ to $\pi/2^k$), then decomposition algorithm (shown in fig 2 1(a)) will have $k$ steps and there w '' be $k$ detail signals

Let us assume that $k = 4$, hence first detail signal will be of length $\cong 512$ i e it can be represented as $d_n^{-9}$, similarly next coarser detail signal can be represented

as $d_n^{-8}$ and so on  Now we plot a graph between $\log_2 L_m$ and $\log_2 P_m$, let the graph obtained be as shown in fig 3 3
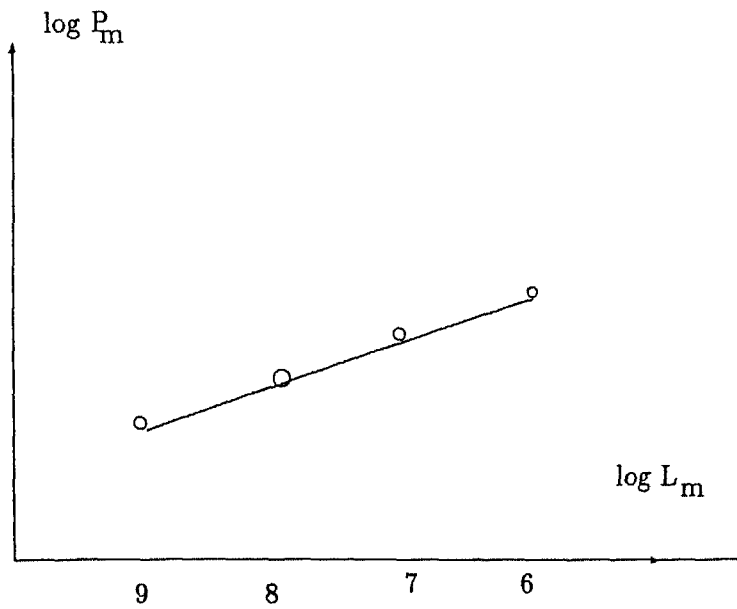


Figure 3 3  Plot between power and length for $\gamma > 1$ case

1 e  coarser detail signals are having more power than finer detail signals, it then is $\gamma > 1$ case and value of $\gamma$ can be obtained from the slope of above curve, let slope be k, then

$\gamma - 1 = k$ and value of $\gamma$ is obtained

If graph obtained is as shown in figure 3 4, then it's a $\gamma = 1$ case

However if graph obtained is as shown in figure 3 5, it is a $\gamma < 1$ case and value of $\gamma$ is $\gamma = 1 - k$, where k is slope of the plot

After obtaining value of $\gamma$ from the slope, $\sigma^2$ will be obtained from eq 3 8

Hence from Constant Q filter bank ( realized by wavelet based decomposition algorithm ) we can get P S D of any type of noise
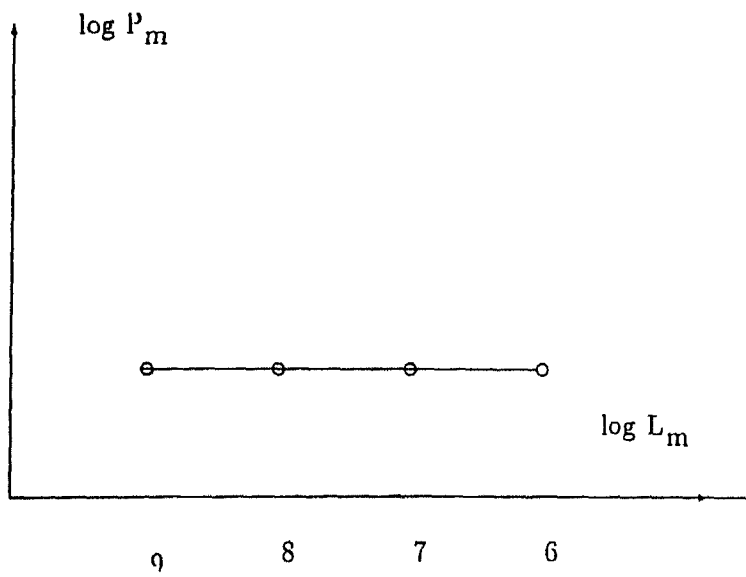
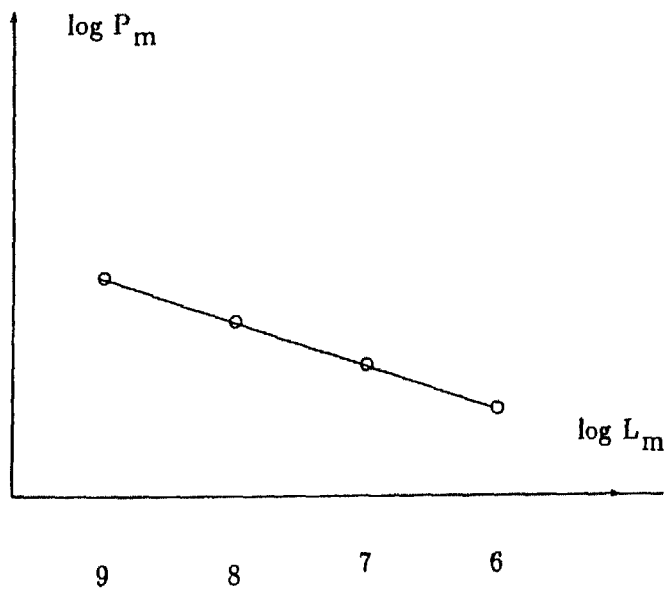Figure 3 4   Plot between power and length for $\gamma = 1$ case



Figure 3 5   Plot between power and length for $\gamma < 1$ case

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# Chapter 4

# Digital filter design

In [9],[10] $1/f$ like spectrum is obtained but that doesn't give an output time series, instead it first generates sequences through a recursive algorithm, then takes auto-correlation of each sequence and desired $1/f$ spectrum is obtained by taking FFT of each autocorrelation and then averaging these FFTs  Since autocorrelation is not a linear operation, we can not get a sequence having $1/f$ like spectrum from this method

In chapter 3 we have discussed that $1/f$ noise can be synthesized from white noise sequences obeying certain power constraints  We will discuss design aspects of these filters in next section

We will also generate $1/f$ noise sequence through digital filters from A/D trans formation using Matched Z transform, for analog to digital transformation with analog filter having a slope of 10db/decade  Designing of this digital filter is dis cussed in section filter 2

## 4 1    filter 1

To realize decomposition ("Constant Q") and reconstruction filters, we have chosen Daubachies [3] 16 pt $h(n)$, reproduced in table 1, as low pass analysis filter  This filter has very high regularity and obeys regularity constraint of theorem 3 1 for all practical cases of $\gamma$s $(2 > \gamma \geq 0)$

| n | h(n) | g(n) |
|---|---|---|
| 0 | 054415842243 | 000117476784 |
| 1 | 312871590914 | 000675449406 |
| 2 | 675630736297 | 000391740373 |
| 3 | 585354683654 | 004870352993 |
| 4 | 015829105256 | 008746094047 |
| 5 | 284015542926 | 013981027917 |
| 6 | 000472484574 | 044088253931 |
| 7 | 128747426620 | 017369301002 |
| 8 | 017369301002 | 128747426620 |
| 9 | 044088253931 | 000472484574 |
| 10 | 013981027917 | 284015542926 |
| 11 | 008746094047 | 015829105256 |
| 12 | 004870352993 | 585354683654 |
| 13 | 000391740373 | 675630736297 |
| 14 | 000675449406 | 312871590914 |
| 15 | 000117476784 | 054415842243 |

Table 4 1   16 pt   h(n) and g(n) used in decomposition and reconstruction algorithm [3]

Any other sequence can be chosen as $h(n)$ if it obeys conditions 1 to 3 of theorem given in appendix A and regularity condition of theorem 3 1

High pass analysis filter $g(n)$ is obtained from $h(n)$ by the relation [3],

$$g(n) = (-1)^n h(-n+1) \tag{4 1}$$

If $h(n)$ is of length 16 i e   $n \in [0,15]$ , $n \in Z$, then $g(n)$ will also be of length 16 with $n \in [-14,1]$ , $n \in Z$   However $g(n)$ is shifted to right so that the shifted version $g'(n)$ will be zero for $n \notin [0,15]$ , $n \in Z$

After choosing $h(n)$ and $g(n)$ as explained above, "Constant Q" filter bank is then realized by writting a computer program for cascade configuration shown in fig  2 1(a)

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To realize reconstruction filter bank of fig 2 1(b) $h(-n)$ and $g(-n)$ are used, however both are again shifted to origin There is one difference between real algorithm written on computer for reconstruction filter bank and fig 2 1(b) This is as follows In computer program a backward shift of 15[1] is made after each summing block of fig 2 1(b) This backward shift is required because of delay caused by $h(-n)$ and $g(-n)$ filters

# 4 2   filter 2

$1/f$ can also be obtained from white noise by passing it through a filter having slope of 10db/decade slope This filter (analog) is realized by pole zero model with poles and zeros are as follows

polel=3 hz                    zerol=8 5hz

pole2=30 hz                   zerol=95hz

pole3=300 hz

slope of this filter will be approx 10db/decade Digital filter is obtained by Matched Z transform for analog to digital transformation with sampling rate of 1000 hz (In Matched Z transform a zero is added at $Z = -1$ if number of zeros are less than number of poles) Difference equation of this filter is

$$\begin{aligned} y(n) &= x(n) - 499x(n-1) - 977x(n-2) + 526x(n-3) \\ &\quad + 1\,961y(n-1) - 1\,087y(n-2) + 123y(n-3) \end{aligned} \qquad (4\,2)$$

A digital filter satisfying above difference equation is realized by writting a computer program

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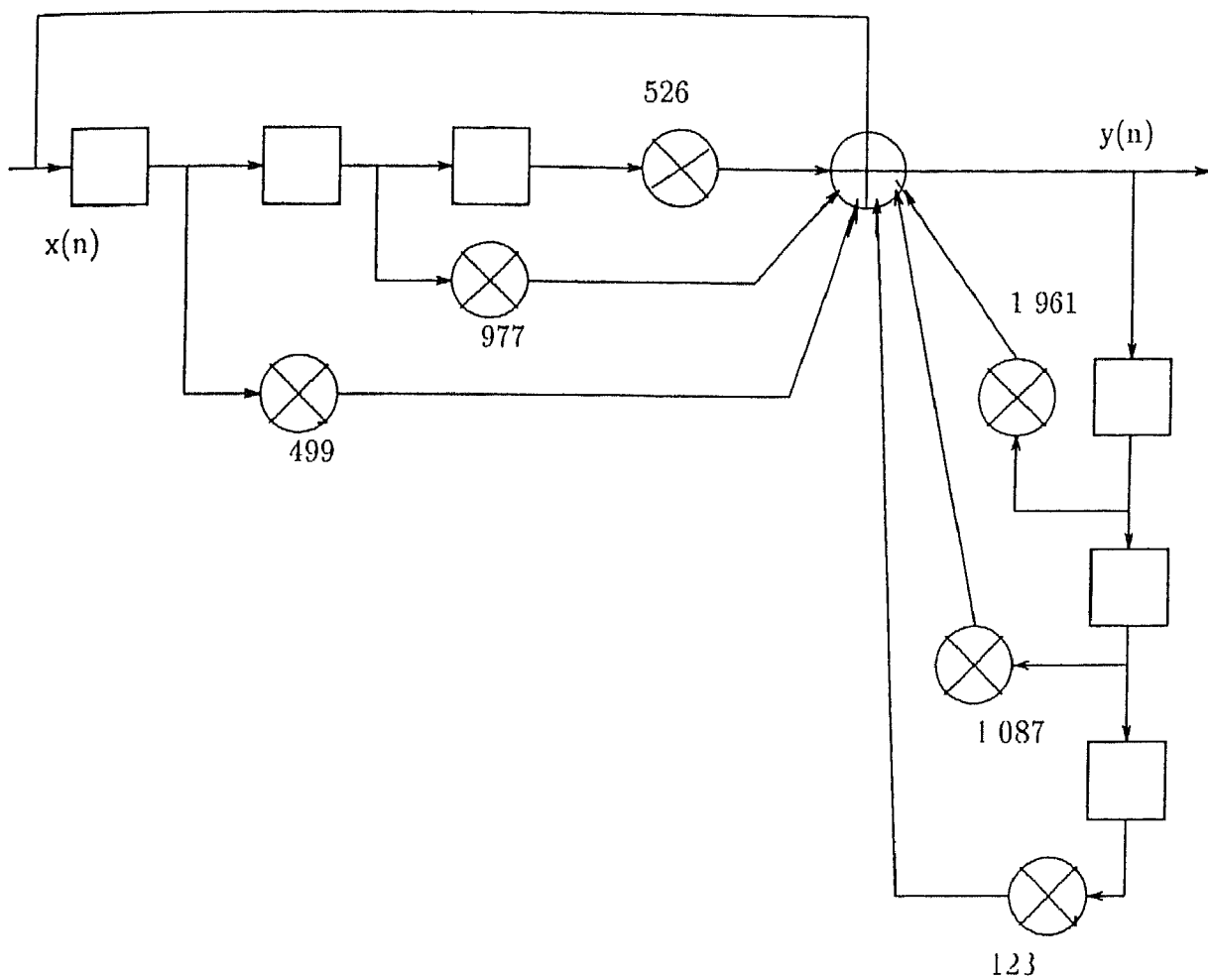[1]Because filter length is 16 If filter length is $l$ then shift should be $l-1$

Figure 4 1 Digital filter through A/D conversion

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# Chapter 5

# Results

In this chapter, the input and output waveforms and power spectral densities of the matched Z and wavelet based filters are given  For the input and output of the matched Z filter  the "detail" and "approximate" decomposition are also given  Furthermore, the synthesis of $1/f$ noise through wavelet based filter bank has been presented

## 5 1   The input white noise sequence

An input white noise sequence has been used for both the filter 1 and filter 2  This sequence has been generated using Knuth s[11] algorithm and its autocorrelation and power spectral density (P S D ) are shown in figs  5 1 and 5 2  respectively  One notes that these are not ideal autocorrelation and ideal P S D  for the sequence to qualify as white noise  but are very good from the point of view of practical realization

## 5 2   The output $1/f$ noise of matched Z filter

1  The P S D  of the output of the matched Z filter  designed in chapter 4 to provide $1/f$ noise at the output, is shown in fig 5 3  Note that there is some deviation of this P S D  from that of ideal $1/f$ noise and this can be attributed

to the nonideality of the input white noise as shown in fig 5 2

2  The time sequence (realization) of the input white noise and the output $1/f$ noise are shown in fig 5 4  A perusal of these plots exhibits the difference between the appearence of the white noise and $1/f$ noise as would be seen say, on an oscilloscope or a recorder

3  A noise sequence can be described in three different ways,

(a) by providing the realization, (b) by providing autocorrelation and P S D or (c) <u>by providing the "detail  and "approximate  decomposition through wavelet analysis</u>  We have carried out the 'detail" and "approximate' decom position of all the waveforms studied in this thesis

The white noise  detail" and "approximate" decomposition is shown in fig  5 5 to exhibit the features of multiresolution analysis based on wavelets  Four blocks (i e four levels of resolution) of fig 2 1(a) have been used  and the 'detail  and 'approximate" output of each block is shown in fig  5 5  The 1/2 'detail' signal corresponds to the output of $(\pi - \pi/2)$ filter, and the 1/2 "approximate  signal corresponds to the output of $(0 - \pi/2)$ filter, these are shown in fig  5 5(a) by red and blue respectively  The 1/4 "detail" corresponds to $(\pi/2 - \pi/4)$ filter output and 1/4 "approximate' to $(0 \quad - \pi/4)$ filter output, and so on  We will return to the considerations of this figure after presenting the 'detail' and  approximate decomposition of $1/f$ noise for comparison

The "detail  and  approximate' decomposition of $1/f$ noise (obtained from the matched Z filter) are shown in fig  5 6  It is interesting to compare the 1/2 "approximate" and 1/2  detail' decomposition of the white and $1/f$ noise as given in fig 5 5(a) and 5 6(a)  respectively  One notes that the 1/2  detail" of $1/f$ noise is very much smaller in magnitude than the 1/2 'approximate  signal  One also notes that whereas the  waveform ratio' of the 'detail" to the  approximate' signals of white noise are the same for all the resolutions, this is not the case for $1/f$ noise  For $1/f$ noise the 'detail" is comparatively small at 1/2 resolution and the 'waveform ratio" of the 'detail  to the "approximate" signal increases with decreasing $(1/4 \rightarrow$

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1/8, → 1/16) resolution The lower resolution corresponds to lower frequency range in P S D , hence fig 5 6 shows that the "detail" signal increases at lower frequencies

The output $1/f$ n ise of the wavelet based filter can also be analyzed in a similar fashion, hence is not presented here

## 5 3 Synthesis of $1/f$ noise through wavelets

In the previous section the detail and "approximate" analysis of $1/f$ noise obtained as output of the matched z filter has been given Now we obtain $1/f$ noise as the output of a wavelet based filter bank (filter 1) shown in fig 2 1(b) The details of this filter are given in chapter 4

The input white noise to this filter is the same as described earlier and is shown in fig 5 7(a) for convenience and comparison The output realization of $1/f$ noise through this filter is shown in fig 5 8

To conclude, we emphasize that whereas the wavelet based filters have the ca pability of both analyzing and synthesizing a required noise waveform, the matched Z filter have only the capability of synthesizing a noise waveform However, the wavelet based filter for synthesis requires the input white noise sequences of differ ent lengths but of same power, whereas the matched Z filter requires only one white noise sequence
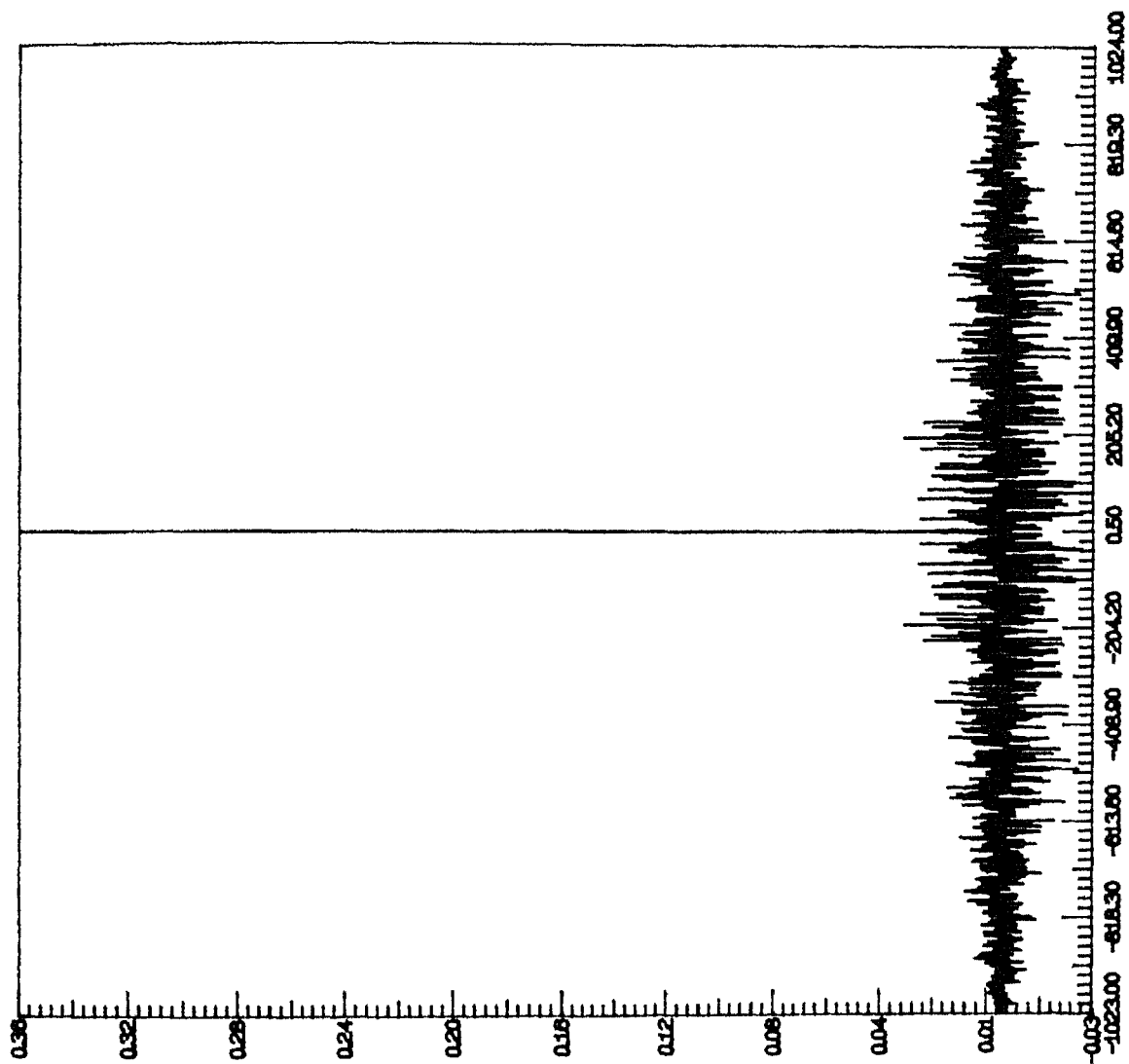
AUTOCORRELATION OF WHITE NOISE SEQUENCE

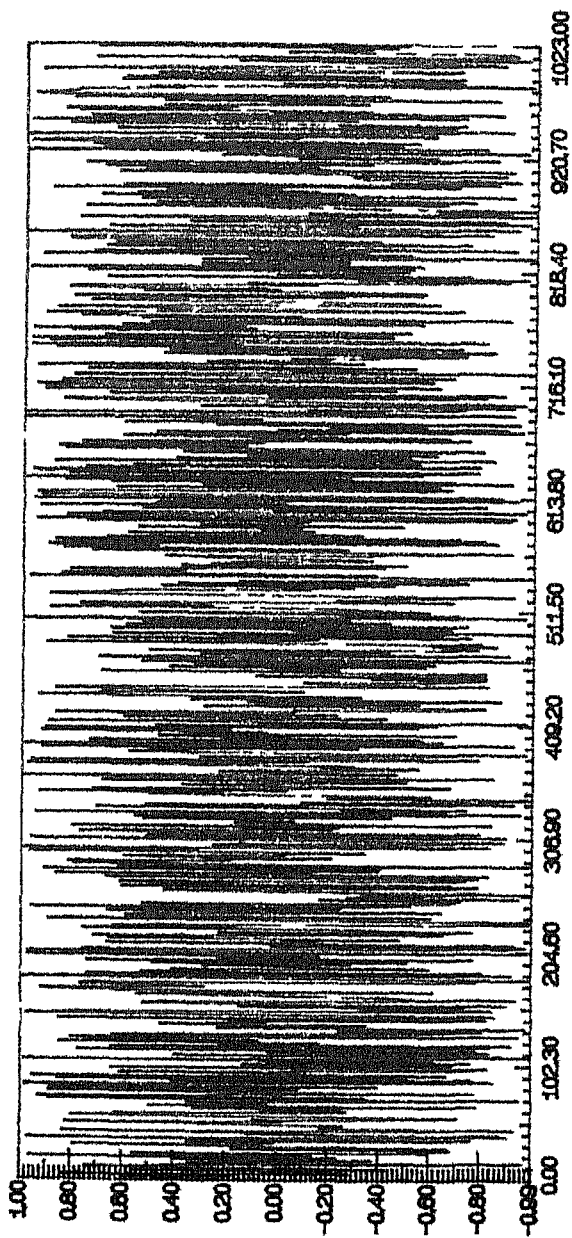* asax * cc-iit k    prithvi * 03 13/95 09·42 55 *

UsPlot

33

PSD OF WHITE NOISE SEQUENCE

*UsPlot*

34

COLOURED NOISE OBTAINED THROUGH FILTER-2

UsPlot

35

# mmmk # cc-iit/k    prithvi # 03/13/95 06:47:40 #
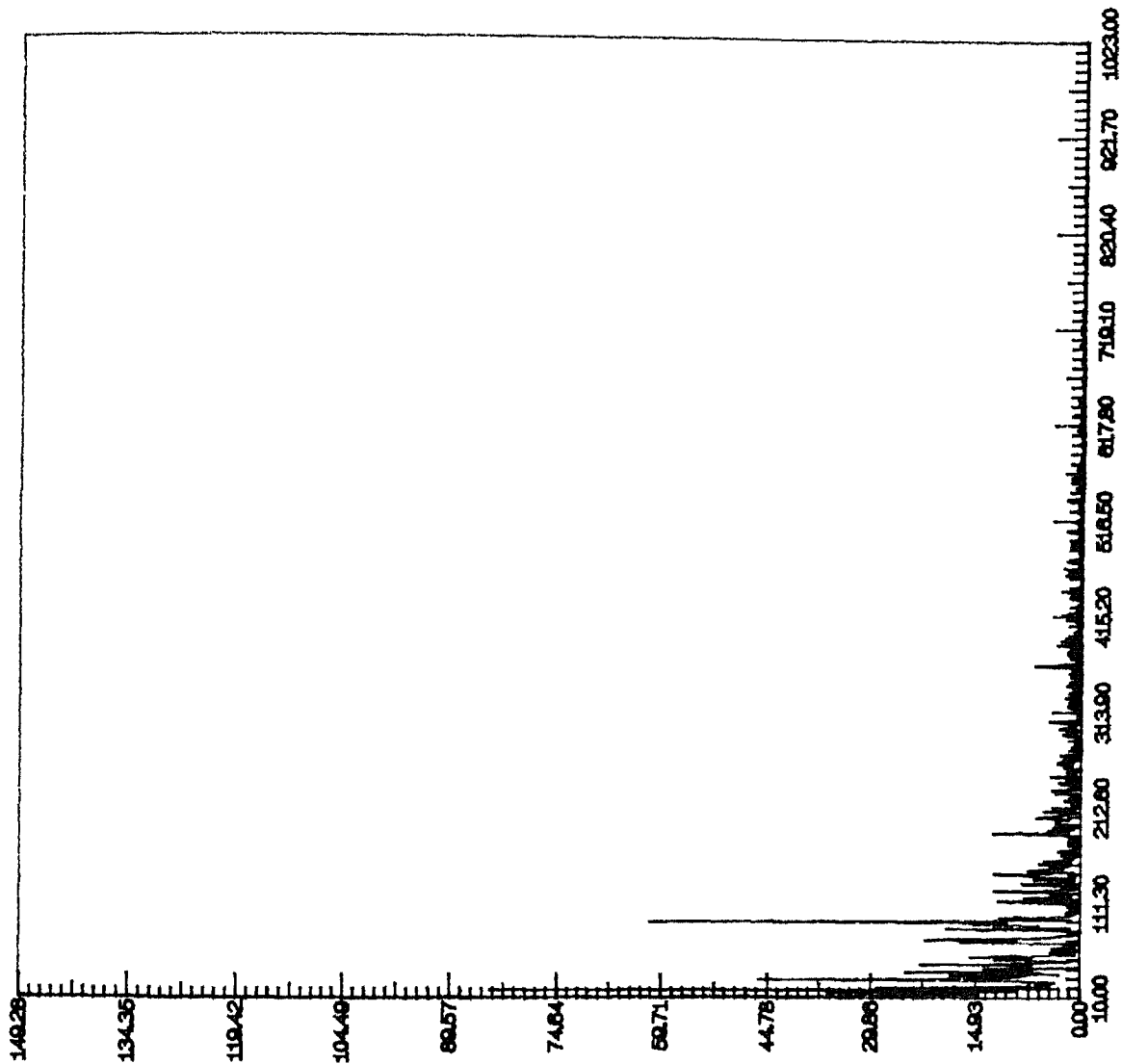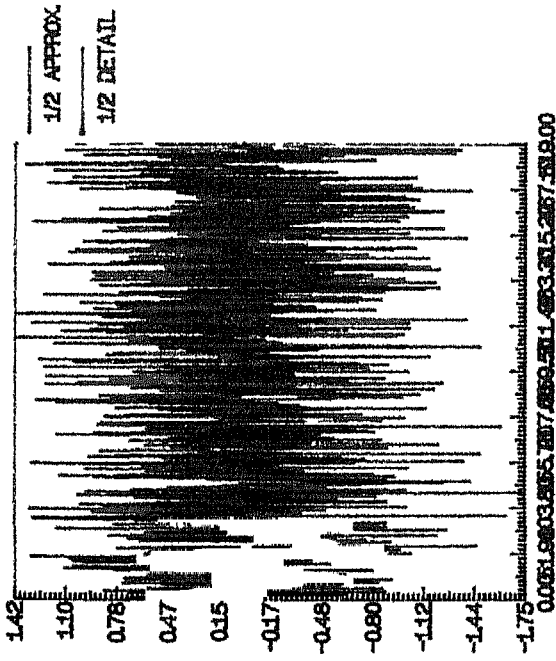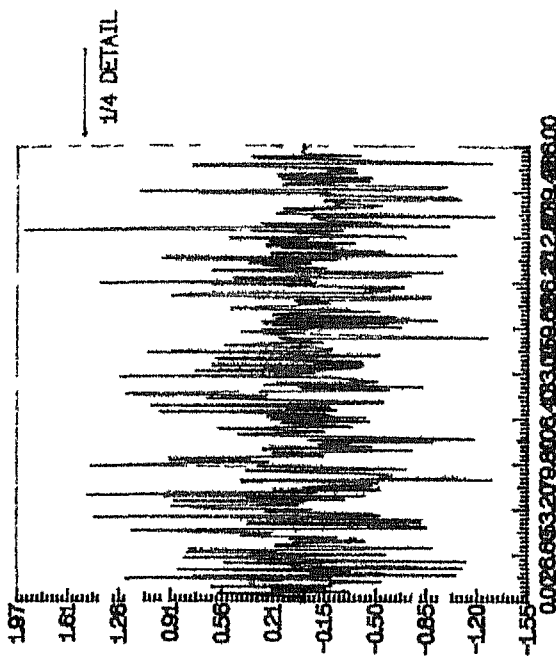
WHITE NOISE SEQUENCE

COLOURED NOISE OBTAINED THROUGH FILTER-2

UsPlot

36

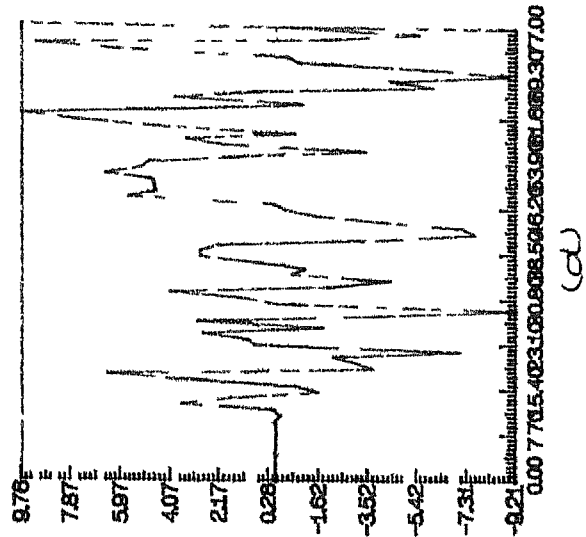# ngpai # cc-iit/k   tejas # 03/13/95 09:39:45 #

WHITE NOISE DECOMPOSITION

(a)

(b)

(c)

(d)

USPlot

* ngpai * cc-iit/k    tejas * 03/13/95 09·58.09 *

USPlot

1/4 DETAIL
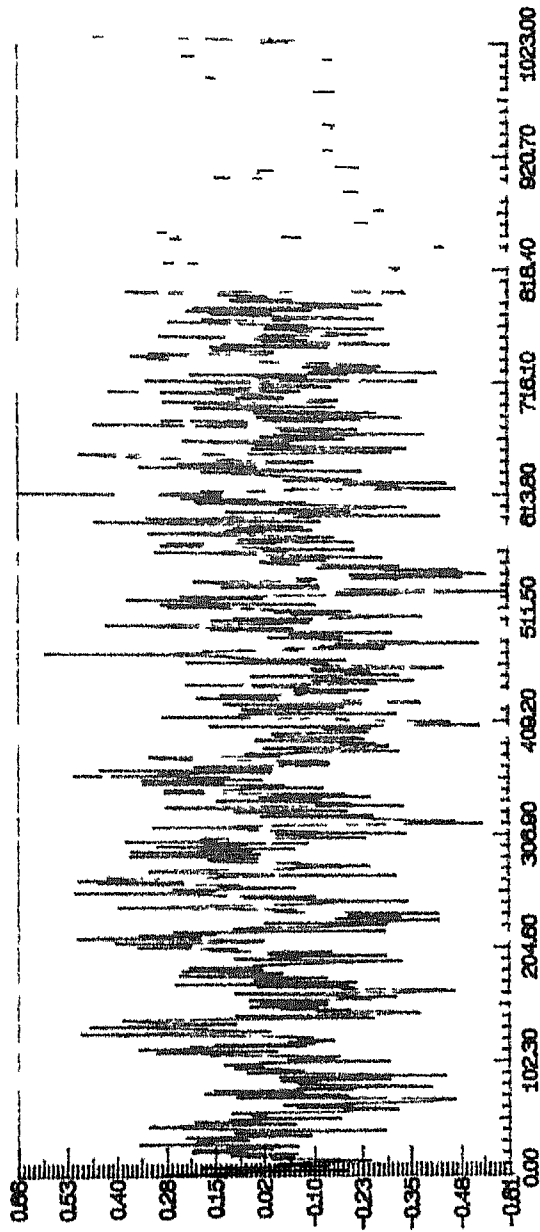
(b)

(d)

1/2 APPROX
1/2 DETAIL

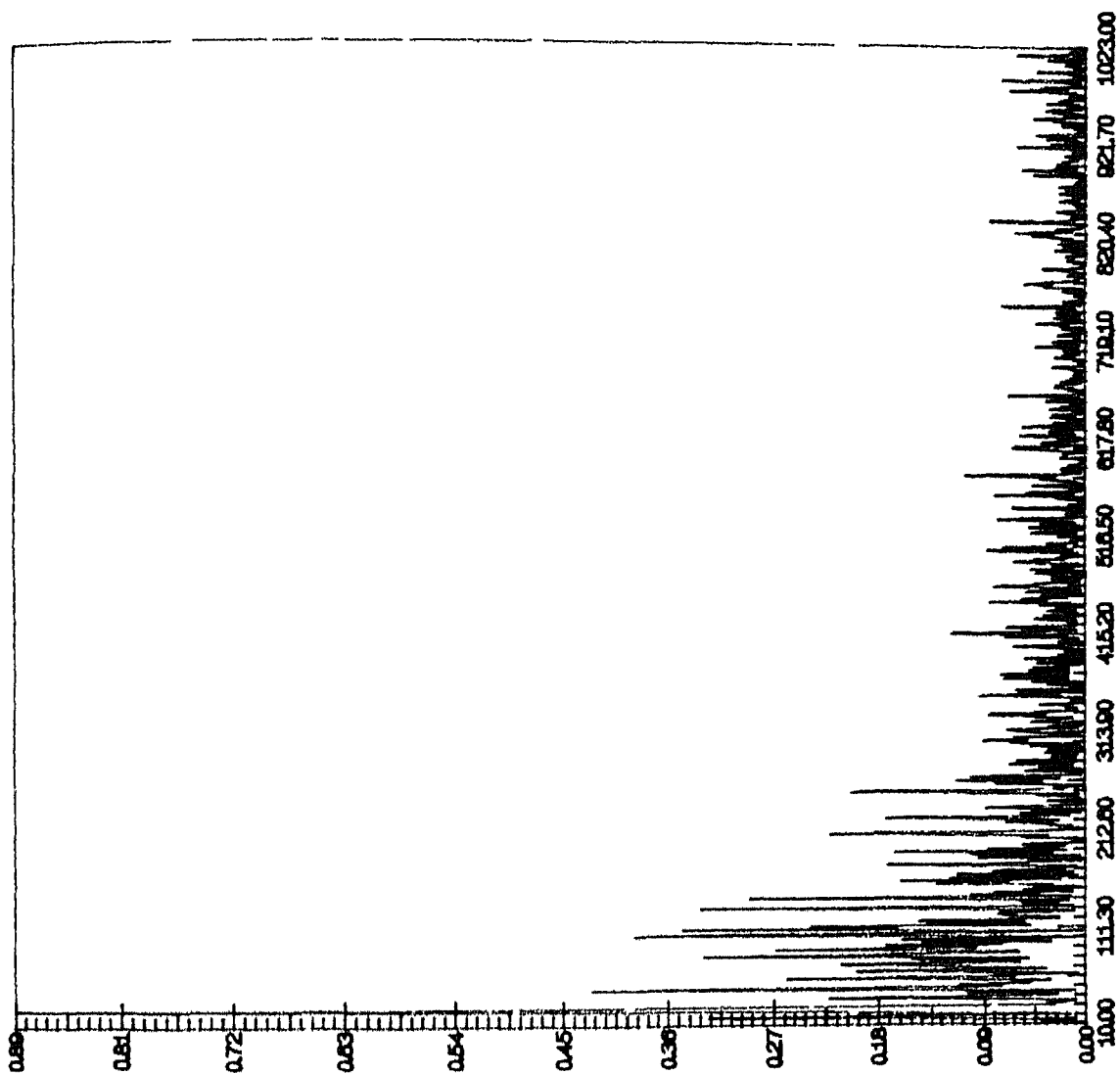1/8 APPROX

(a)

(c)
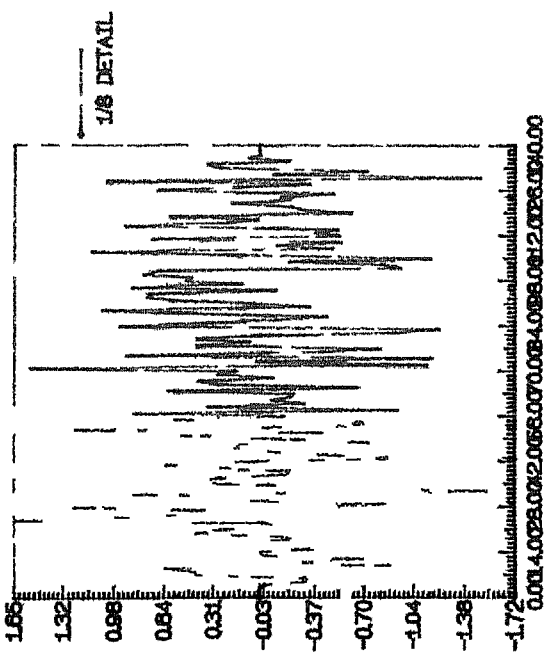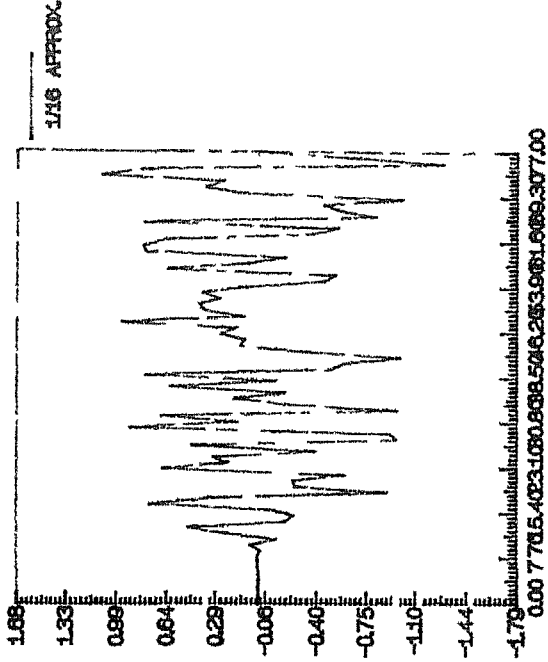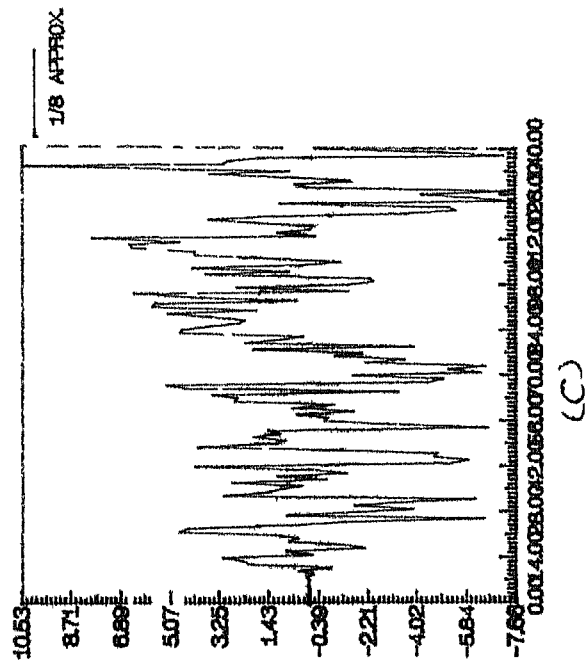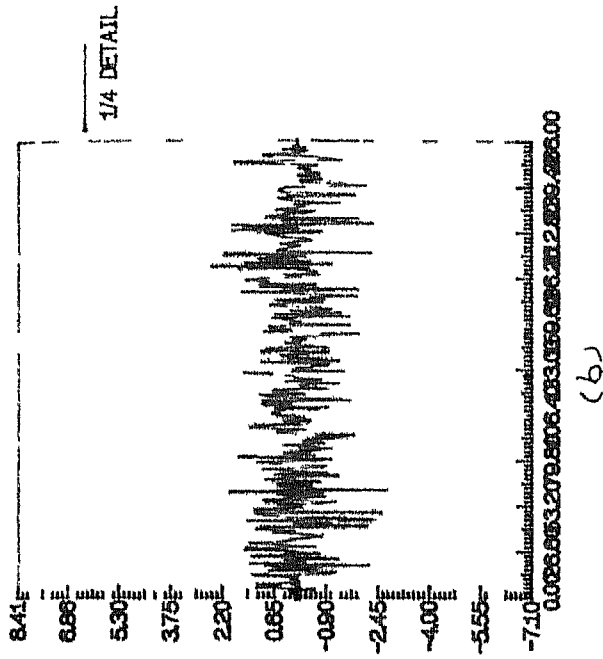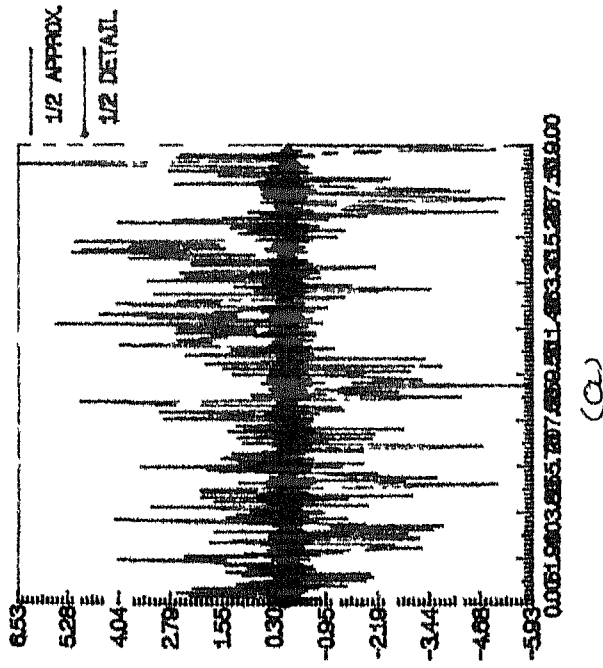
COLOUR NOISE DECOMPOSITION
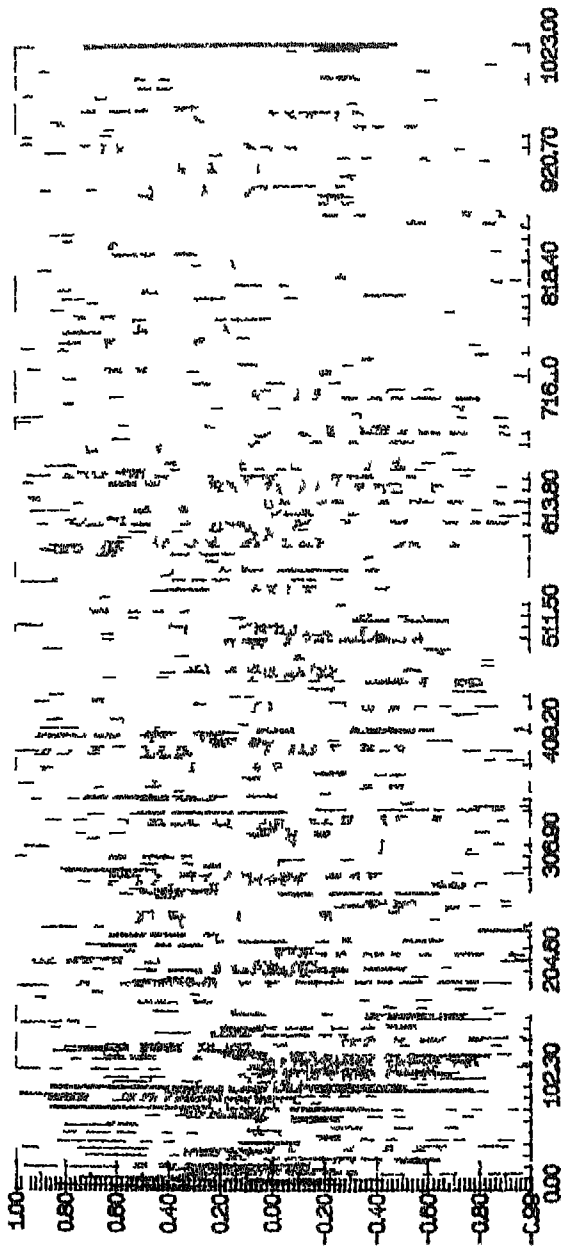
WHITE NOISE SEQUENCE(S) USED IN FILTER-1

1/f NOISE OBTAINED FROM FILTER-1

USPlot

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PSD OF 1/f NOISE OBTAINED THROUGH FILTER-2

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UsPlot

# Chapter 6

# Conclusion

In the present work measurement of low frequency noise through "constant Q" fil
tering is done  All cases of low frequency noise are studied

Simulation of low frequency noise has been done by two methods  (1) Matched Z
transform technique for analog to digital conversion with analog filter having slope
of 10 db/decade, and (2) Wavelet transform based technique  It has been found
that these techniques give $1/ \mid f \mid^{\gamma}$ noise when white noise is applied to the input

It has been found that time for measurement of noise through "Constant Q"
filter is more than that required in FFT method in which first autocorrelation of
sequence and then it's FFT is taken to get P S D of sequence  However with this
method modelling of 1/f noise is possible  This is the greatest advantage of this
method

# Appendix A

We know that decomposition and reconstruction is achieved through filters $h(n)$ and $g(n)$ where

$$h(n) = <\phi_{10}, \phi_{0n}> \qquad \text{(A 1)}$$

$$g(n) = <\psi_{10}, \phi_{0n}> \qquad \text{(A 2)}$$

Thus to construct $h(n)$ and $g(n)$ we have to start with $\phi$ satisfying 2 1, 2 2, 2 3, 2 4, 2 5, and then we get $h(n)$ and $g(n)$ from A 1 and A 2 respectively However we can get $h(n)$ and $g(n)$ directly without bothering about $\phi$ This is due to following theorem [3]

**THEOREM   A 1** *Let there be a sequence such that,*

*1* $\sum_n | h(n) | | n |^\epsilon < \propto$ *for some* $\epsilon > 0$

*2* $\sum_n h(n-2k)h(n-2l) = \delta_{kl}$

*3* $\sum_n h(n) = 2^{1/2}$

   *suppose also that* $m_0(\varepsilon) = 2^{-1/2} \sum_n h(n)e^{jn\epsilon}$

   *can be written as* $m_0(\varepsilon) = [\frac{1}{2}(1 + e^{j\epsilon})]^N [\sum_n f(n)e^{jn\epsilon}]$

   *where*

*4* $\sum_n | f(n) | | n |^\epsilon < \propto$ *for some* $\epsilon > 0$

*5* $Sup_{\epsilon \in R} | \sum_n f(n)e^{jn\epsilon} | < 2^{N-1}$

*Define*

$$g(n) = (-1)^n h(-n+1) \qquad \text{(A 3)}$$

$$\Phi(\varepsilon) = (2\pi)^{-1/2} \prod_{j=1}^{\infty} m_0(2^{-j}\varepsilon) \qquad \text{(A 4)}$$

$$\psi(x) = 2^{1/2} \sum_n g(n)\phi(2x-n) \qquad \text{(A 5)}$$

*Where $\Phi(\varepsilon)$ is fourier transform of $\phi(x)$*

*Then $\phi_{jk}(x) = 2^{-j/2}\phi(2^{-j}x-n)$ define a multiresolution analysis,the $\phi_{jk}$ are the associated orthonormal wavelet basis*

Hence we can choose $h(n)$ directly satisfying first three properties of above theorem, and can apply multiresolution analysis on any signal

# A 1   Regularity

A function f(x) is $R^{th}$ order regular if its fourier transform $F(\omega)$ decays according to

$$\Gamma(\omega) \sim \Re(\mid \omega \mid^{-R}), \quad \omega \to \infty$$

Where the notation $\Re(\ )$ is to be understood to mean that

$lim_{\omega \to \infty} \frac{F(\omega)}{|\omega|^{-R}} < \infty$

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